Elementary Row Operation

I. Interchange two row

II. Multiply a row by a nonzero real number

III. Replace a row by its sum with a multiple of another row.

(I) ~~Determinants~~ changes the sign of the determinant.

(II) have the effect of multiplying the value of determinant by the nonzero real number.

(III) not change the value of the determinant.

$AX = b$, $(A|b)$ augmented matrix

reduced row echelon form if (i) the matrix is in row echelon form (ii) The first nonzero entry in each row is the only nonzero entry in its column.

$$\left(\begin{array}{ccccc|c} 1 & 0 & \times & 0 & 0 & 0 \\ 0 & 1 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_4 &= 0 \\ x_5 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \\ x_4 &= 0 \\ x_5 &= 0 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3.$$

1,2,4,5 pivot column

x_1, x_2, x_4, x_5 lead variable

x_3 no pivot .. free variable.

齐次·参数~~解~~形式

通解

strict triangular form

例1: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right)$. $\quad x_1 = 1 - 2x_2 - x_3 = 1 - 8 - 2 = -9$, $\quad x_2 = 2 + x_3 = 4$, $\quad x_3 = 2$

唯一解

$$\left(\begin{array}{cccccc|c} * & 0 & 0 & \alpha & & \cdot & b_1 \\ 0 & * & 0 & \alpha & & & b_2 \\ 0 & 0 & * & \alpha & \times & \alpha & \vdots \\ \vdots & & & & & & \\ 0 & & 0 & 0 & 0 & & b_n \end{array}\right).$$

$0 + 0 + \dots 0 \neq b_n \neq 0$ 无解.

例3

$$\begin{cases} x_1+x_2+x_3+x_4+x_5=2 \\ x_1+x_2+x_3+2x_4+2x_5=3 \\ x_1+x_2+x_3+2x_4+3x_5=2. \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1&1&1&1&1&2 \\ 1&1&1&2&2&3 \\ 1&1&1&2&3&2 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1&1&1&0&0&1 \\ 0&0&0&1&0&2 \\ 0&0&0&0&1&-1 \end{array}\right)$$

$$\begin{cases} x_1+x_2+x_3=1 \\ x_4=2 \\ x_5=-1 \end{cases} \quad \begin{cases} x_1=1-x_2-x_3 \\ x_4=2 \\ x_5=-1 \end{cases} \quad \begin{cases} x_1=1-x_2-x_3 \\ x_2=x_2 \\ x_3=x_3 \\ x_4=2 \\ x_5=-1 \end{cases}$$

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5 \end{pmatrix} = \begin{pmatrix} 1\\0\\0\\2\\-1 \end{pmatrix} + \begin{pmatrix} -1\\1\\0\\0\\0 \end{pmatrix} x_2 + \begin{pmatrix} -1\\0\\1\\0\\0 \end{pmatrix} x_3$$

$$\left.\begin{array}{l} AX_1=b \\ AX_2=b \end{array}\right\} \Rightarrow A(X_1-X_2)=0, \quad AX_1=0, \ AX_2=0, \quad A(k_1X_1+k_2X_2)=0$$

identity matrix

$I=\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ $AI=IA=A$. A is nonsingular or invertible if exists a matrix B such that $AB=BA=I$. $A^{-1}=B$, $B^{-1}=A$.

$(A|I)\to(I|A^{-1})$

$(AB)^{-1}=B^{-1}A^{-1}$, A transpose. A^T, if $A^T=A$ then A symmetric 对称

$(AB)^T=B^TA^T$. $(A^T)^T=A$ $A^T=-A$ A 反对称 skew.

$(\alpha A)^T=\alpha A^T$

$(A+B)^T=A^T+B^T$

A 列线性无关 A^TA 可逆.

A is nonsingular/invertible

$\Leftrightarrow$ $AX=0$. 只有0解. $\Leftrightarrow$ A is row equivalent I.

$\Leftrightarrow$ $AX=b$ 唯一解. $X=A^{-1}b$.

$$A=\begin{bmatrix} 2&4&2\\1&5&2\\4&-1&9 \end{bmatrix} = \overset{L}{\begin{bmatrix} 1&0&0\\ \frac{1}{2}&1&0\\ 2&-3&1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2&4&2\\0&3&1\\0&0&8 \end{bmatrix}}$$

$AB=(Ab_1, Ab_2, \dots, Ab_n)$, $X=\begin{bmatrix} 3&1\\2&4\\1&2 \end{bmatrix}$ $Y=\begin{bmatrix} 1&2\\2&4\\3&1 \end{bmatrix}$

$$AB=\begin{pmatrix} \alpha_1^T B \\ \alpha_2^T B \\ \vdots \\ \alpha_n^T B \end{pmatrix}. \quad XY^T=\begin{bmatrix} 3&1\\2&4\\1&2 \end{bmatrix}\begin{bmatrix} 1&2&3\\2&4&1 \end{bmatrix} = \begin{bmatrix} 3\times1+1\times2 & 3\times2+1\times4 & 3\times3+1\times1 \\ 2\times1+2\times4 & 2\times2+4\times4 & 2\times3+4\times1 \\ 1\times1+2\times2 & 1\times2+2\times4 & 1\times3+2\times1 \end{bmatrix}$$

(I) if $x \in V$ and α is a scalar, $\alpha x \in V$. V is space.

(II) if $x, y \in V$, then $x+y \in V$.

If S is a nonempty number of a vector space V; and S satisfies the condition (I) $\alpha x \in S$ whenever $x \in S$ for any scalar α. (II). $x+y \in S$ whenever $x \in S$ and $y \in S$. then S is said to be a subspace of V. (空间和子空间定义)

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the sum of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ is called a linear combination of $v_1, v_2, \cdots v_n$. the set of all linear combination of $v_1, v_2, \cdots v_n$ is called the span of $v_1, v_2, \cdots v_n$, $\text{span}\{v_1, v_2, \cdots v_n\}$

if $v_1, v_2, \cdots v_n$ are element of a vector space V then $\text{span}\{v_1, v_2 \cdots v_n\}$ is subspace of V. (线性组合, span (生成) 之定义)

~~the~~ if $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ implies that all the scalars $c_1, c_2, \cdots c_n$ must equal 0 then $v_1, v_2, \cdots v_n$ linear independent (线性无关)

if there exist scalar $c_1, c_2, \cdots c_n$ not all zero, such that $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ then vector $v_1, v_2, \cdots v_n$ linearly dependent. (线性相关)

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the vector $v_1, v_2, \cdots v_n$ form a basis for V. if and only if.

(i) $v_1, v_2, \cdots v_n$ are linearly independent (ii) $v_1, v_2 \cdots v_n$ span V. (basis 基之定义).

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if V has a basis consisting of n vector, ~~we~~ then V has dimmen n, the subspace $\{0\}$ of V, have dimension 0 (dimension 维定义)

If V is a vector space of dimension n then (I) any set of n linearly independent vector span V; (II) any n vector that span V are linearly independent. (维性质)

A mapping L from a vector space V into a vector space W is said to be a linear transformation if $L(c_1v_1+c_2v_2)=c_1L(v_1)+c_2L(v_2)$ for all $v_1,v_2\in V$ and for all scalar c_1,c_2。（线性变换）

Given a linear mapping $g: V\to W$. with $\dim V=n$ and $\dim W=m$

g is injective (单射) $\Longleftrightarrow \operatorname{rank}(g)=n \Longleftrightarrow \ker(g)=\{0\}$, g is surjective (满射) $\Longleftrightarrow \operatorname{rank}(g)=m \Longleftrightarrow \operatorname{im}(g)=R^m$ (像)

g is bijective (双射) $\Longleftrightarrow \operatorname{rank}(g)=m=n$ (秩)

核 $\ker(L)=\{v\in V \mid L(v)=0_W\}$ (if $L: V\to W$.)

像 $L(S)=\{w\in W \mid w=L(v),\ \text{for some } v\in S\}$

the image of the entire vector space $L(V)$ is called the range of L

(i) $\ker(L)$ is a subspace of V (ii) $L(S)$ is a subspace of W

$L(x)=Ax$, $A=[L(e_1), L(e_2), \cdots L(e_n)]$ 变换阵

If $E=\{v_1,v_2,\cdots v_n\}$ and $F=\{w_1,w_2,\cdots w_n\}$ are ordered based for vector space V and W respectively, then corresponding to each linear transformation $L: V\to W$, there is an $m\times n$ matix A such that $[L(v)]_F=A[v]_E$, $A=[\,[L(v_j)]_F\,]$.

$x=[v]_E$. (the coordinate vector of v with respect to E)

$y=[w]_F$ (the coordinate vector of w with respect to F)

then L maps v into w if and only if A map x into y.

$x \xrightarrow{A} y$

$\beta \to \beta'$

$M=C_B A P_B$

if (i) B is the matrix representing L with respect $[u_1, u_2]$ (ii) A is the matrix representing with respect $[e_1, e_2]$ (iii) U is the transition matrix corresponding to the change of basis from $[u_1, u_2]$ to $[e_1, e_2]$ then

$B = U^{-1}AU$, $U = [u_1, u_2]$

Let A and B be $n \times n$ matrices, B is said to be similar to A if there exist a nonsingular matrix S such that $B = S^{-1}AS$.

$\det(A) = \det(B)$ 特征多项式相同 特征值相同

$\mathrm{trace}(A) = \mathrm{trace}(B)$ 迹相同, 迹=对角线元素和.

Let A be $n \times n$ matrix and λ be a scalar. The following statements are equivalent (a) λ is an eigenvalue of A. (b) $(A - \lambda I)X = 0$ has a nontrivial solution (c) $N(A - \lambda I) \neq \{0\}$ (d) $A - \lambda I$ is singular (e) $\det(A - \lambda I) = 0$.

$A_{n\times n}$ is said to be diagonalizable if there exists a nonsingular matrix X and a diagonal matrix D such that $X^{-1}AX = D$.

$A_{n\times n}$ is diagonalizable if and only if A has n linearly independent eigenvector.

$AX_i = \lambda_i X_i$, $D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$ eigenvalue, $X = [X_1, X_2, X_3]$ eigenvector.

A 可对角化 $\Longleftrightarrow \begin{cases} n\text{个互不相同特征值.} \\ \text{实对称.} \\ \text{有}n\text{个线性无关特征向量.} \end{cases}$ $A = XDX^{-1}$

$A^k = XD^kX^{-1}$.

scalar Product ~~dot~~. $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ $X^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ $\langle x, y \rangle \equiv x \cdot y$

$\|X\| = \sqrt{X^T X} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ the distance between X and Y is defined to be the number $\|X - Y\|$, $X^T y = \|X\|\|Y\| \cos\theta$,

$\cos\theta = \frac{X^T Y}{\|X\|\|Y\|}$. forming unit vector. $u = \frac{1}{\|X\|} X$, $\|u\| = 1$

X and Y are said to be orthogonal if $X^T y = 0$

Scalar projection of X and y $\alpha = \frac{X^T y}{\|y\|}$

Vector projection of X and y $p = \alpha u = \alpha \frac{1}{\|y\|} y = \frac{X^T y}{y^T y} y$

Two subspace X and Y of R^n are said to be orthogonal if $x^T y = 0$ for every $x \in X$ and every $y \in Y$, $X \perp Y$

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to solve the least squares problem $AX = b$. we must solve $A^T A X = A^T b$, $\hat{X} = (A^T A)^{-1} A^T b$, $\hat{X}$ is the unique least squares solution of the system $AX = b$ $\hat{y} = A\hat{X}$

Inner Product (i) $\langle X, X \rangle \geq 0$ with equality if and only if $X = 0$

(ii) $\langle X, y \rangle = \langle y, X \rangle$ for all X and y in V.

(iii) $\langle \alpha X + \beta y, z \rangle = \alpha \langle X, z \rangle + \beta \langle y, z \rangle$

if $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$ then $\{v_1, v_2, \cdots v_n\}$ is an orthogonal set of vectors. An orthonormal set of vector is an orthogonal set of unit vectors. 正交阵 $Q = [q_1\ q_2 \cdots q_n]$ $Q^T = Q^{-1}$ $\|QX\| = \|X\|$